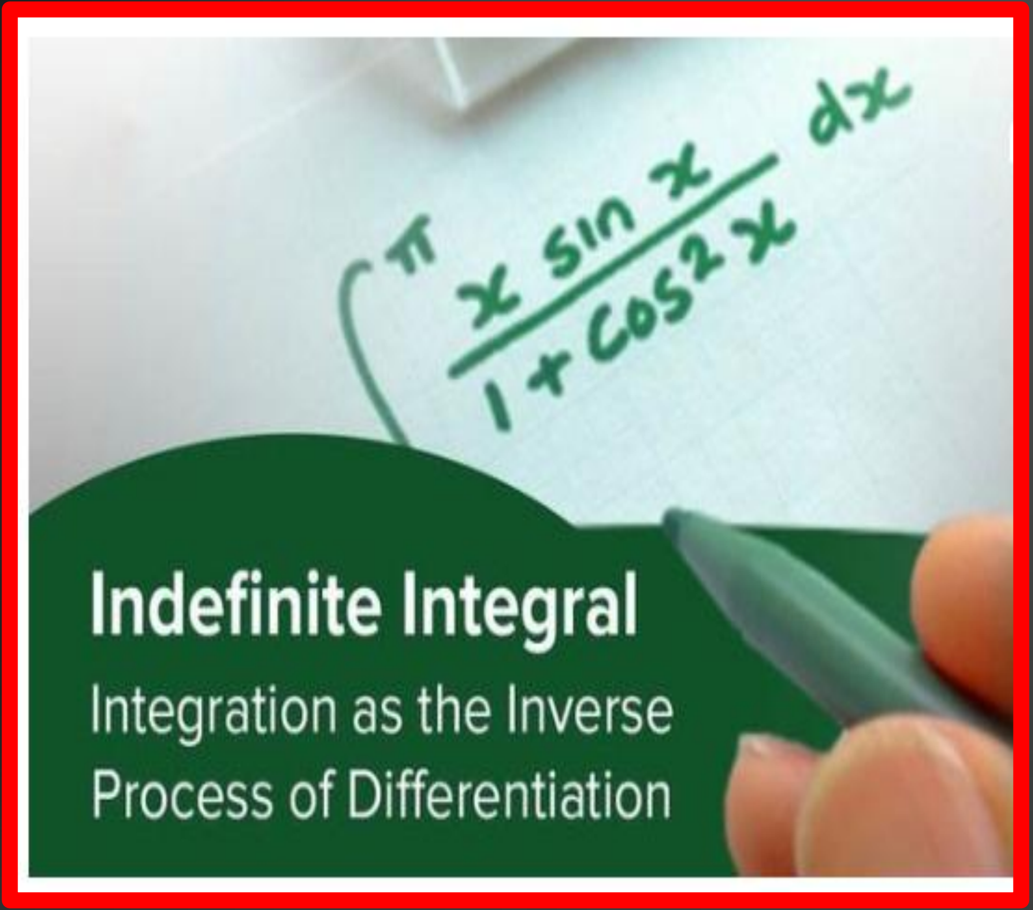


$$\int k\,dx = kx + C$$

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C$$

I S M

INTEGRALS – MODULE 4

Substitution Method for Some Important Integrals of Trigonometric Functions

- $\int \tan x \, dx = \log|\sec x| + C$

We know that tan x = sin x / cos x. Therefore,

$\int \tan x \, dx = \int (\sin x / \cos x) \, dx.$

Now, let's substitute cos x = t, so that sin x dx = – dt. Therefore,

$\int \tan x \, dx = -\int (dt / t) = -\log|\cos x| + C$

Or, $\int \tan x \, dx = \log|\sec x| + C$

$$-\log|\cos x| = \log|\cos x|^{-1} = \log\left|\frac{1}{\cos x}\right|$$

- $\int \sec x \, dx = \log|\sec x + \tan x| + C$

On multiplying both the numerator and denominator by (sec x + tan x), we have

$\int \sec x \, dx = \int \{\sec x (\sec x + \tan x) \, dx\} / (\sec x + \tan x)$

Now, let's substitute (sec x + tan x) = t, so that $\sec x \tan x + \sec^2 x = dt$

That is, $\sec x (\sec x + \tan x) \, dx = dt$.

Therefore, $\int \sec x \, dx = \int (dt / t) = \log|t| + C = \log|\sec x + \tan x| + C$

Similarly, we can prove

$\int \cot x \, dx = \log|\sin x| + C$

Similarly, we can prove

$\int \operatorname{cosec} x \, dx = \log|\operatorname{cosec} x - \cot x| + C$

LET'S RECALL……TRIGONOMETRIC IDENTITIES AND FORMULAE

Angle sum and difference identities

$$\sin(x+y) = \sin x \cos y + \cos x \sin y$$

$$\sin(x-y) = \sin x \cos y - \cos x \sin y$$

$$\cos(x+y) = \cos x \cos y - \sin x \sin y$$

$$\cos(x-y) = \cos x \cos y + \sin x \sin y$$

$$\tan(x+y) = \frac{\tan x + \tan y}{1 - \tan x \tan y}$$

$$\tan(x-y) = \frac{\tan x - \tan y}{1 + \tan x \tan y}$$

Sum Identities (Sum to Product Identities)

$$\cos x + \cos y = 2\cos\frac{x+y}{2}\cos\frac{x-y}{2}$$

$$\cos x - \cos y = -2\sin\frac{x+y}{2}\sin\frac{x-y}{2}$$

$$\sin x + \sin y = 2\sin\frac{x+y}{2}\cos\frac{x-y}{2}$$

$$\sin x - \sin y = 2\cos\frac{x+y}{2}\sin\frac{x-y}{2}$$

Half Angle Identities

$$\sin x = 2\sin\frac{x}{2}\cos\frac{x}{2} = \frac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}}$$

$$\cos x = \cos^2\frac{x}{2} - \sin^2\frac{x}{2} = 1 - 2\sin^2\frac{x}{2} = 2\cos^2\frac{x}{2} - 1 = \frac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}}$$

$$\tan x = \frac{2\tan\frac{x}{2}}{1-\tan^2\frac{x}{2}}$$

$$\sin\frac{x}{2} = \pm\sqrt{\frac{1-\cos x}{2}}$$

$$\cos\frac{x}{2} = \pm\sqrt{\frac{1+\cos x}{2}}$$

Triple Angle Formulas

$$\sin 3x = 3\sin x - 4\sin^3 x$$

$$\cos 3x = 4\cos^3 x - 3\cos x$$

$$\tan 3x = \frac{3\tan x - \tan^3 x}{1 - 3\tan^2 x}$$

Product Identities (Product to Sum Identities)

$2\cos x\ \cos y = \cos(x+y) + \cos(x-y)$

$-2\sin x\ \sin y = \cos(x+y) - \cos(x-y)$

$2\sin x\ \cos y = \sin(x+y) + \sin(x-y)$

$2\cos x\ \sin y = \sin(x+y) - \sin(x-y)$

INTEGRATION USING TRIGONOMETRIC IDENTITIES……….

Ex 7.3, 1

Find the integral of $\sin^2(2x+5)$

$$\int \sin^2(2x+5)\,dx = \int \frac{1-\cos 2(2x+5)}{2}\,dx$$

$$= \frac{1}{2}\int 1-\cos(4x+10)\,dx$$

$$= \frac{1}{2}\left[\int 1\,dx - \int \cos(4x+10)\,dx\right]$$

$$= \frac{1}{2}\left[x - \frac{\sin(4x+10)}{4} + C\right]$$

$$= \frac{x}{2} - \frac{1}{8}\sin(4x+10) + C$$

We know that

$$\cos 2\theta = 1 - 2\sin^2\theta$$

$$2\sin^2\theta = 1 - \cos 2\theta$$

$$\sin^2\theta = \frac{1}{2}[1-\cos 2\theta]$$

Replace θ by $(2x+5)$

$$\sin^2(2x+5) = \frac{1-\cos 2(2x+5)}{2}$$

(As $\int \cos(ax+b)\,dx = \frac{\sin(ax+b)}{a} + C$)

Ex 7.3, 3

Integrate the function - cos 2x cos 4x cos 6x

We know that

$$2\cos A\cos B = [\cos(A+B) + \cos(A-B)]$$

Replace A by 2x & B by 4x

$$2\cos 2x\cos 4x = \cos(2x+4x) + \cos(2x-4x)$$

$$2\cos 2x\cos 4x = \cos 6x + \cos 2x \qquad (\because \cos(-x) = \cos x)$$

$$\cos 2x\cos 4x = \frac{1}{2}(\cos 6x + \cos 2x)$$

$$\int(\cos 2x\cos 4x\cos 6x)\,dx$$

$$= \int\left(\frac{1}{2}(\cos 6x + \cos 2x)\cos 6x\right)dx$$

$$= \frac{1}{2}\left[\int(\cos 6x)^2\,dx + \int\cos 2x\,.\cos 6x\,dx\right]$$

Solving both these integrals separately

I S M

$\int(\cos^2 6x)$

We know that

$$\cos^2\theta = \frac{\cos 2\theta + 1}{2}$$

Replace θ by $6x$

$$\cos^2 6x = \frac{\cos 12x + 1}{2}$$

$$\int \cos^2 6x \, dx$$

$$= \frac{1}{2}\int(\cos 12x + 1)\ dx$$

$\int \cos 2x \cos 6x \, dx$

We know that

$$2\cos A \cos B = \cos(A+B) + \cos(A-B)$$

$$\cos A \cos B = \frac{1}{2}[\cos(A+B) + \cos(A-B)$$

Replace A by 2x & B by 6x

$$\cos 2x \cos 6x$$

$$= \frac{1}{2}[\cos(2x+6x) + \cos(2x-6x)$$

$$= \frac{1}{2}[\cos 8x + \cos 4x]\, dx$$

$$\int \cos 2x \cos 6x \, dx$$

$$= \frac{1}{2}\int(\cos 8x + \cos 4x)\, dx$$

$$\int(\cos 2x \cos 4x \cos 6x)\, dx$$

$$= \frac{1}{2}\left[\frac{1}{2}\int(\cos 12x + 1)\, dx + \frac{1}{2}\int(\cos 8x + \cos 4x)\, dx\right]$$

$$= \frac{1}{4}\left[\int \cos 12x \, dx + \int 1\, dx + \int \cos 8x\, dx + \int \cos 4x \, dx\right]$$

$$= \frac{1}{4}\left[\frac{\sin 12x}{12} + x + \frac{\sin 8x}{8} + \frac{\sin 4x}{4}\right] + C$$

ISM

Ex 7.3, 9

Integrate $\frac{\cos x}{1+\cos x}$

$$\int \frac{\cos x}{1+\cos x}\, dx$$

$$= \int \left(\frac{\cos x + 1 - 1}{1+\cos x}\right) dx$$

$$= \int \left(\frac{1 + \cos x - 1}{1+\cos x}\right) dx$$

$$= \int \left(\frac{1+\cos x}{1+\cos x} - \frac{1}{1+\cos x}\right) dx$$

$$= \int 1 - \frac{1}{1+\cos x}\, dx$$

$$= \int 1\, dx - \int \frac{1}{1+\cos x}\, dx$$

$$= \int 1\, dx - \int \frac{1}{2\cos^2\frac{x}{2}}\, dx$$

$1 + \cos x = 2\cos^2\frac{x}{2}$

$$= \int 1\, dx - \int \frac{1}{2}\sec^2\frac{x}{2}\, dx$$

$$= \int 1\, dx - \frac{1}{2}\int \sec^2\frac{x}{2}\, dx$$

$$= x - \frac{1}{2}\frac{\tan\frac{x}{2}}{\frac{1}{2}} + C$$

$\int \sec^2(ax+b)\, dx = \frac{\tan(ax+b)}{a} + C$

$$= x - \frac{2}{2}\tan\frac{x}{2} + C$$

$$= x - \tan\frac{x}{2} + C$$

ISM

Ex 7.3, 13

Integrate the function $\frac{\cos 2x - \cos 2\alpha}{\cos x - \cos \alpha}$

$$\int \frac{\cos 2x - \cos 2\alpha}{\cos x - \cos \alpha}\ dx$$

$$= \int \frac{(2\cos^2 x - 1) - (2\cos^2 \alpha - 1)}{\cos x - \cos \alpha}\ dx \quad (\cos 2\theta = 2\cos^2\theta - 1)$$

$$= \int \frac{2\cos^2 x - 1 - 2\cos^2 \alpha + 1}{\cos x - \cos \alpha}\ dx$$

$$= \int \frac{2\cos^2 x - 2\cos^2 \alpha + 1 - 1}{\cos x - \cos \alpha}\ dx$$

$$= \int \frac{2(\cos^2 x - \cos^2 \alpha)}{\cos x - \cos \alpha}\ dx$$

$$= 2\int \frac{(\cos x - \cos \alpha)(\cos x + \cos \alpha)}{(\cos x - \cos \alpha)}\ dx$$

$$= 2\int (\cos x + \cos \alpha)\ dx$$

$$= 2\left(\int \cos x\, dx + \int \cos \alpha\, dx\right)$$

$$= 2\left(\int \cos x\, dx + \cos \alpha \int 1.dx\right)$$

$$= 2(\sin x + x\cos \alpha) + C$$

ISM

Ex 7.3, 16

$\int \tan^4 x\,dx$

$\int \tan^4 x\,dx = \int \tan^2 x \,.\, \tan^2 x\,dx$

$= \int (\sec^2 x - 1) \tan^2 x\,dx$

We know that

$\tan^2\theta = \sec^2\theta - 1$

$= \int (\sec^2 x \,.\, \tan^2 x - \tan^2 x)\,dx$

$= \int \tan^2 x \,.\, \sec^2 x\,dx - \int \tan^2 x\,dx$

Solving both these integrals separately

$\int \tan^2 x \,.\, \sec^2 x\,dx$

Let $\tan x = t$

$\sec^2 x = \frac{dt}{dx}$

$\sec^2 x \;\; dx = dt$

Now,

$\int \tan^2 x \,.\, \sec^2 x \,.\, dx$

$= \int t^2 .\, dt \;= \frac{t^3}{3} + C$

$= \frac{\tan^3 x}{3} + C_1$

$\int \tan^2 x\,dx$

$= \int (\sec^2 x - 1)\,dx$

$= \int \sec^2 x\,dx - \int 1 .\, dx$

As $\int x^n\,dx = \frac{x^{n+1}}{n+1} + C$

& $\int \sec^2 x\;dx = \tan x + C$

$= \tan x - x + C_2$

Now, $\int \tan^4 x\,dx = \int \tan^2 x \,.\, \sec^2 x\,dx - \int \tan^2 x\;dx$

$= \frac{\tan^3 x}{3} + C_1 - (\tan x - x + C_2)$

$= \frac{\tan^3 x}{3} - \tan x + x + C$ *(Where $C = C_1 - C_2$)*

ISM

Ex 7.3, 20

Integrate the function $\frac{\cos 2x}{(\cos x + \sin x)^2}$

$$\int \frac{\cos 2x}{(\cos x + \sin x)^2}$$

$$= \int \frac{\cos^2 x - \sin^2 x}{(\cos x + \sin x)^2}\, dx \qquad (\cos 2\theta = \cos^2\theta - \sin^2\theta)$$

$$= \int \frac{(\cos x - \sin x)(\cos x + \sin x)}{(\cos x + \sin x)^2}\, dx$$

$$= \int \frac{\cos x - \sin x}{\cos x + \sin x}\, dx$$

Let $\cos x + \sin x = t$

Differentiating w.r.t. x

$$-\sin x + \cos x = \frac{dt}{dx}$$

$$(\cos x - \sin x)dx = dt$$

Thus, our equation becomes

$$= \int \frac{1}{t}\, dt$$

$$= \log|t| + C$$

$$= \boldsymbol{\log|\cos x + \sin x| + C}$$

I S M

Ex 7.3, 22

$$\frac{1}{\cos(x-a)\cos(x-b)}$$

$$\int \frac{1}{\cos(x-a)\cos(x-b)}$$

Multiply & Divide by $sin(a-b)$

$$= \int \frac{\sin(a-b)}{\sin(a-b)} \times \frac{1}{\cos(x-a)\cos(x-b)}\, dx$$

$$= \frac{1}{\sin(a-b)} \int \frac{\sin(a-b)}{\cos(x-a)\cos(x-b)}\, dx$$

$$= \frac{1}{\sin(a-b)} \int \frac{\sin(a-b+x-x)}{\cos(x-a)\cos(x-b)}\, dx$$

$$= \frac{1}{\sin(a-b)} \int \frac{\sin((x-b)+(a-x))}{\cos(x-a)\cos(x-b)}\, dx$$

I S M

$$= \frac{1}{\sin(a-b)} \int \frac{\sin((x-b)-(x-a))}{\cos(x-a)\cos(x-b)}\, dx$$

Using $\sin(A-B) = \sin A \cos B - \cos A \sin B$

Replace A by $(x-b)$ *& B by* $(x-b)$

$$\sin((x-b)-(x-a)) = \sin(x-b)\cos(x-a) - \cos(x-b)\sin(x-a)$$

$$= \frac{1}{\sin(a-b)} \int \frac{\sin(x-b)\cos(x-a) - \cos(x-b)\sin(x-a)}{\cos(x-a)\cos(x-b)}\, dx$$

$$= \frac{1}{\sin(a-b)} \int \left(\frac{\sin(x-b)\cos(x-a)}{\cos(x-a)\cos(x-b)} - \frac{\cos(x-b)\sin(x-a)}{\cos(x-a)\cos(x-b)}\right) dx$$

$$= \frac{1}{\sin(a-b)} \left[\int \left(\frac{\sin(x-b)}{\cos(x-b)} - \frac{\sin(x-a)}{\cos(x-a)}\right) dx\right]$$

$$= \frac{1}{\sin(a-b)} \left[\int \tan(x-b) - \tan(x-a)\ dx\right]$$

$$= \frac{1}{\sin(a-b)} \left[\int \tan(x-b)\ dx - \int \tan(x-a)\, dx\right]$$

ISM

Using $\int \tan x \, dx = -\log|\cos x| + C$

$$= \frac{1}{\sin(a-b)}\left[-\log|\cos(x-b)| + \log|\cos(x-a)|\right] + C$$

$$= \frac{1}{\sin(a-b)}\left[\log|\cos(x-a)| - \log|\cos(x-b)|\right] + C$$

$$= \frac{1}{\sin(a-b)} \log\left|\frac{\cos(x-a)}{\cos(x-b)}\right| + C$$

HOME ASSIGNMENT

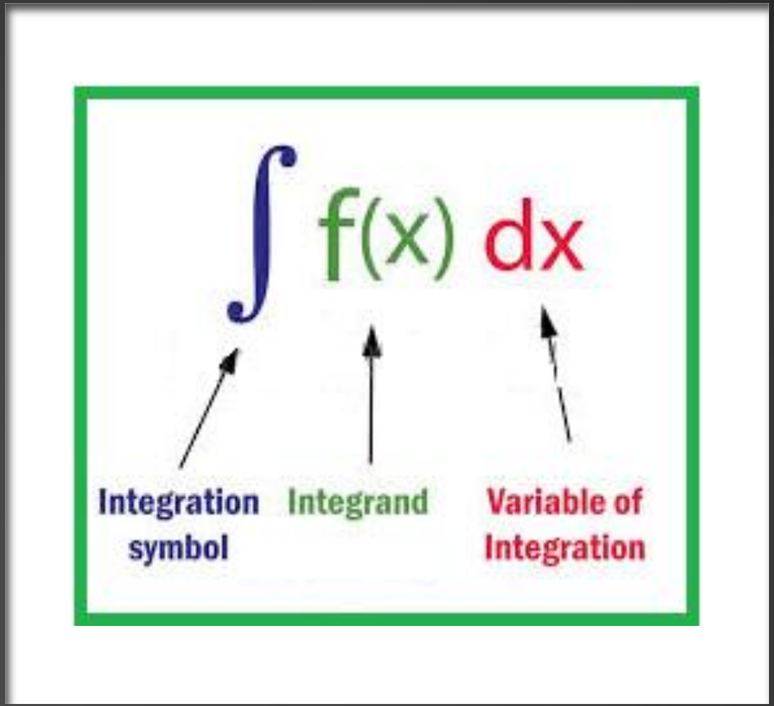
EXERCISE – 7.3

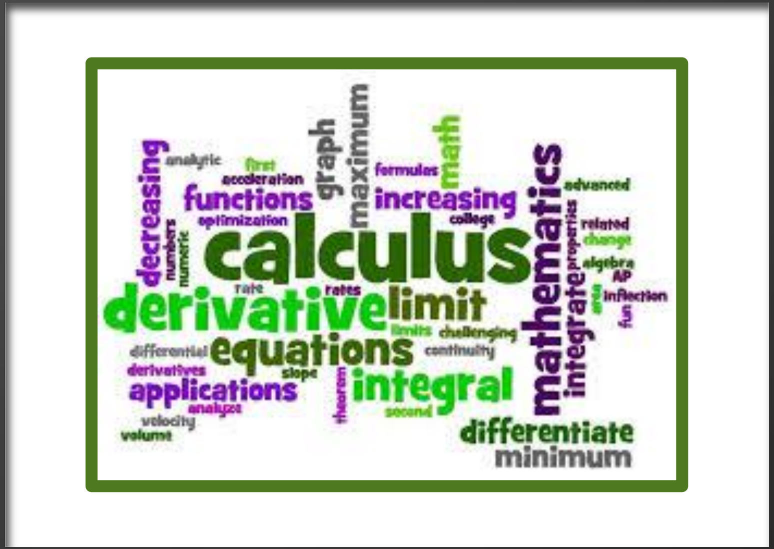
Q. NO – 6,8,10,15,17,24

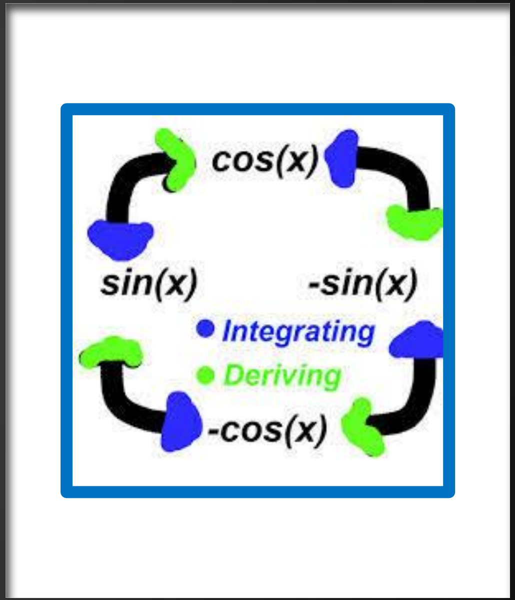
I S M

INTEGRALS

MODULE - 5







INTEGRATION OF SOME PARTICULAR FUNCTIONS

Integrals of some special functions

1. $\int \frac{dx}{x^2 - a^2} = \frac{1}{2a} \log \left| \frac{x-a}{x+a} \right| + c$

2. $\int \frac{dx}{a^2 - x^2} = \frac{1}{2a} \log \left| \frac{a+x}{a-x} \right| + c$

3. $\int \frac{dx}{x^2 + a^2} = \frac{1}{a} \tan^{-1} \frac{x}{a} + c$

4. $\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1} \frac{x}{a} + c$

5. $\int \frac{dx}{\sqrt{x^2 - a^2}} = \log \left| x + \sqrt{x^2 - a^2} \right| + C$

6. $\int \frac{dx}{\sqrt{x^2 + a^2}} = \log \left| x + \sqrt{x^2 + a^2} \right| + C$

Ex 7.4, 1

$\frac{3x^2}{x^6+1}$

$$\int \frac{3x^2}{x^6+1}\,dx = \int \frac{3x^2}{(x^3)^2+1}\,dx$$

Let $x^3 = t$

Diff both sides w.r.t.x

$$3x^2\,dx = dt$$

$$\therefore \int \frac{3x^2}{x^6+1}\,dx = \int \frac{dt}{t^2+1}$$

$$= \int \frac{dt}{t^2+(1)^2}$$

It is of form

$$\int \frac{dt}{x^2+a^2} = \frac{1}{a}\tan^{-1}\frac{x}{a} + C$$

$$= \tan^{-1}(t) + C$$

$$= \boldsymbol{\tan^{-1}(x^3) + C}$$

Ex 7.4, 2

$\frac{1}{\sqrt{1+4x^2}}$

$$\int \frac{1}{\sqrt{1+4x^2}}\,dx = \int \frac{1}{\sqrt{4\left(\frac{1}{4}+x^2\right)}}\,.dx$$ *(Taking 4 common)*

$$= \frac{1}{2}\int \frac{1}{\sqrt{x^2+\frac{1}{4}}}\,.\,dx$$

$$= \frac{1}{2}\int \frac{1}{\sqrt{x^2+\left(\frac{1}{2}\right)^2}}\,.\,dx$$

It is of form

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \log\left|x+\sqrt{x^2+a^2}\right| + C$$

$$= \frac{1}{2}\left[\log\left|x+\sqrt{x^2+\frac{1}{4}}\right|\right] + C$$

$$= \frac{1}{2}\log\left|x+\sqrt{\frac{4x^2+1}{4}}\right| + C$$

$$= \boldsymbol{\frac{1}{2}\log\left|2x+\sqrt{1+4x^2}\right| + C}$$

Ex 7.4, 4

$$\frac{1}{\sqrt{9-25x^2}}$$

$$\int \frac{1}{\sqrt{9-25x^2}}\,dx$$

$$= \int \frac{1}{\sqrt{25\left(\frac{9}{25}-x^2\right)}}\,dx$$ *(Taking 25 common)*

$$= \frac{1}{5}\int \frac{1}{\sqrt{\frac{9}{25}-x^2}}\,dx$$

It is of form

$$\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\frac{x}{a} + C$$

$$= \frac{1}{5}\left[\sin^{-1}\frac{x}{\frac{3}{5}} + C\right]$$

$$= \frac{1}{5}\,\sin^{-1}\frac{5x}{3} + C$$

Ex 7.4, 6

$$\frac{x^2}{1-x^6}$$

$$\int \frac{x^2}{1-x^6}\,dx = \int \frac{x^2}{1-(x^3)^2}\,dx$$

Now, Try this......

Answer $= \frac{1}{6}\,log\left|\frac{1+x^3}{1-x^3}\right| + C$

Ex 7.4, 9

$$\frac{\sec^2 x}{\sqrt{\tan^2 x + 4}}$$

Let $\tan x = t$

Diff both sides w.r.t. x

$$\sec^2 x \; dx = dt$$

It is of form

$$\int \frac{dx}{\sqrt{x^2 + a^2}} = \log\left|x + \sqrt{x^2 + a^2}\right| + C$$

$$\therefore \int \frac{\sec^2 x}{\sqrt{\tan^2 x + 4}} = \int \frac{1}{\sqrt{t^2 + (2)^2}} \cdot dt$$

$$= \log\left|t + \sqrt{t^2 + (2)^2}\right| + C$$

$$= \log\left|\tan x + \sqrt{\tan^2 x + 4}\right| + C$$

SOLVE THE FOLLOWING :

EX – 7.4

Q.NO: 3,5,7,8.

IMPORTANT FORMS TO BE CONVERTED INTO SPECIAL INTEGRALS

(i) Form I → $\int \frac{1}{ax^2+bx+c}\,dx$ or $\int \frac{1}{\sqrt{ax^2+bx+c}}\,dx$

Express ax^2+bx+c as sum or difference of two squares *i.e.*,

$$ax^2+bx+c = a\left[x^2+\frac{b}{a}x+\frac{c}{a}\right] = a\left[\left(x+\frac{b}{2a}\right)^2+\left(\frac{c}{a}-\frac{b^2}{4a^2}\right)\right]$$

We find the integral reduced to the form $\frac{1}{a}\int \frac{dt}{t^2 \pm k^2}$ and hence evaluate.

Example: Find $\int \frac{dx}{9x^2+6x+5}$

$(a+b)^2$

$$9x^2+6x+5 = 9\left[x^2+\frac{6}{9}x+\frac{5}{9}\right] = 9\left[x^2+2.x.\frac{1}{3}+\left(\frac{1}{3}\right)^2+\frac{5}{9}-\left(\frac{1}{3}\right)^2\right]$$

$$= 9\left[\left(x+\frac{1}{3}\right)^2+\left(\frac{2}{3}\right)^2\right]$$

$$\therefore \int \frac{dx}{9x^2+6x+5} = \frac{1}{9}\times\frac{1}{2/3}\tan^{-1}\left(\frac{x+1/3}{2/3}\right)+C = \frac{1}{6}\tan^{-1}\left(\frac{3x+1}{2}\right)+C$$

<u>Alternate method</u>

Since $9x^2$ is a perfect square so can be written as $(3x)^2$

$$9x^2+6x+5 = (3x)^2+2(3x)(1)+1+4$$

$$= (3x+1)^2+2^2$$

$$\therefore \int \frac{dx}{9x^2+6x+5} = \int \frac{dx}{(3x+1)^2+2^2}$$

$$= \frac{1}{2\times3}\tan^{-1}\left(\frac{3x+1}{2}\right)+C$$

(ii) Form II → $\int \frac{px+q}{ax^2+bx+c}\,dx$ or $\int \frac{px+q}{\sqrt{ax^2+bx+c}}\,dx$

Put $px + q = A\frac{d}{dx}(ax^2 + bx + c) + B$

The numerator is split into two parts, the first contains A(the differentiation of the quadratic) and the other is a constant B(free of x) . Now A and B can be found out by equating the coefficient of each term of the above expression on LHS and RHS.

Example: Find $\int \frac{5x-2}{3x^2+2x+1}dx$

Write $5x - 2 = A(6x + 2) + B$

Comparing the coefficients of x,

we get $5 = 6A \Rightarrow A = 5/6$

Comparing the constants,

we get $-2 = 2A + B$

$\Rightarrow -2 = 2\left(\frac{5}{6}\right) + B \Rightarrow B = -11/3$

$\therefore \quad 5x - 2 = \frac{5}{6}(6x + 2) - \frac{11}{3}$

Now, $\int \frac{5x-2}{3x^2+2x+1}dx = \int \frac{5/6(6x+2)-11/3}{3x^2+2x+1}dx$

$= \frac{5}{6}\int \frac{6x+2}{3x^2+2x+1}dx - \frac{11}{3}\int \frac{dx}{3x^2+2x+1}$

Bcoz numerator is diff of denominator so use substitution method to integrate

Change the denominator using completing square method and use $\int \frac{dx}{x^2+a^2}$ to integrate

Ex 7.4, 14

$$\frac{1}{\sqrt{8+3x-x^2}}$$

$$\int \frac{1}{\sqrt{8+3x-x^2}}\,dx$$

$$= \int \frac{1}{\sqrt{8-(x^2-3x)}}\,dx$$

$$= \int \frac{1}{\sqrt{8-\left[x^2-2(x)\left(\frac{3}{2}\right)\right]}}\,dx$$

$$= \int \frac{1}{\sqrt{8-\left[x^2-2(x)\left(\frac{3}{2}\right)+\left(\frac{3}{2}\right)^2-\left(\frac{3}{2}\right)^2\right]}}\,dx$$ *(Adding and subtracting $\left(\frac{3}{2}\right)^2$)*

$$= \int \frac{1}{\sqrt{8-\left[\left(x-\frac{3}{2}\right)^2-\left(\frac{3}{2}\right)^2\right]}}\,dx$$

$$= \int \frac{1}{\sqrt{8-\left(x-\frac{3}{2}\right)^2+\frac{9}{4}}}\,dx$$

COMPLETING THE SQUARE METHOD

$$= \int \frac{1}{\sqrt{8+\frac{9}{4}-\left(x-\frac{3}{2}\right)^2}}\,dx$$

$$= \int \frac{1}{\sqrt{\frac{41}{4}-\left(x-\frac{3}{2}\right)^2}}\,dx$$

$$= \int \frac{1}{\sqrt{\left(\frac{\sqrt{41}}{2}\right)^2-\left(x-\frac{3}{2}\right)^2}}\,dx$$

It is of form

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a}+C$$

$$= \sin^{-1}\left(\frac{x-\frac{3}{2}}{\frac{\sqrt{41}}{2}}\right)+C$$

$$= \sin^{-1}\left(\frac{2x-3}{\sqrt{41}}\right)+C$$

$\int \frac{dx}{\sqrt{9x-4x^2}}$ equals

A. $\frac{1}{9}\sin^{-1}\left(\frac{9x-8}{8}\right)+C$

B. $\frac{1}{9}\sin^{-1}\left(\frac{8x-9}{9}\right)+C$

C. $\frac{1}{3}\sin^{-1}\left(\frac{9x-8}{8}\right)+C$

D. $\frac{1}{2}\sin^{-1}\left(\frac{9x-8}{8}\right)+C$

$$\int \frac{dx}{\sqrt{9x-4x^2}} = \int \frac{dx}{\sqrt{-4\left(x^2-\frac{9}{4}x\right)}}$$ *(Taking −4 common)*

$$= \int \frac{dx}{\sqrt{-4\left(x^2-2(x)\left(\frac{9}{8}\right)\right)}}$$

$$= \int \frac{dx}{\sqrt{-4\left[x^2-2(x)\left(\frac{9}{8}\right)+\left(\frac{9}{8}\right)^2-\left(\frac{9}{8}\right)^2\right]}}$$

COMPLETE THE SUM……

I S M

Ex 7.4, 19

Integrate $\frac{6x+7}{\sqrt{(x-5)(x-4)}}$

$$\int \frac{6x+7}{\sqrt{(x-5)(x-4)}} \, . \, dx$$

$$= \int \frac{6x+7}{\sqrt{x^2-9x+20}} \, . \, dx$$

$6x + 7 = A(2x-9) + B$

Comparing coefficients of x,

$2A = 6 \Rightarrow A = 3$

Comparing constants,

$-9A + B = 7 \Rightarrow -9(3)+B = 7 \Rightarrow B = 34$

$$\therefore \int \frac{6x+7}{\sqrt{x^2-9x+20}} dx = 3\underbrace{\int \frac{2x-9}{\sqrt{(x^2-9x+20)}} dx}_{I_1} + 34 \underbrace{\int \frac{dx}{\sqrt{(x^2-9x+20)}}}_{I_2} \quad ...(1)$$

Rough

$(x^2 - 9x + 20)' = 2x - 9$

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$$I_1 = 3\int \frac{2x-9}{\sqrt{(x^2-9x+20)}} . dx$$

Let $x^2 - 9x + 20 = t$

$(2x-9)dx = dt$

$$= 3\int \frac{1}{\sqrt{t}} . dt \quad = 3\int (t)^{\frac{-1}{2}} . dt$$

$$= 3\frac{t^{\frac{1}{2}}}{\frac{1}{2}} + C_1 \quad = 6\ t^{\frac{1}{2}} + C_1$$

$$I_1 = 6\sqrt{x^2 - 9x + 20} + C_1$$

$$I_2 = 34\int \frac{1}{\sqrt{x^2-9x+20}} . dx$$

$$= 34\int \frac{1}{\sqrt{x^2 - 2(x)\left(\frac{9}{2}\right) + \left(\frac{9}{2}\right)^2 - \left(\frac{9}{2}\right)^2 + 20}} . dx$$

$$= 34\int \frac{1}{\sqrt{\left(x-\frac{9}{2}\right)^2 - \left(\frac{9}{2}\right)^2 + 20}} . dx$$

$$= 34\int \frac{1}{\sqrt{\left(x-\frac{9}{2}\right)^2 - \left(\frac{1}{2}\right)^2}} . dx$$

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$$= 34\left[log\left|x - \frac{9}{2} + \sqrt{\left(x - \frac{9}{2}\right)^2 + \left(\frac{1}{2}\right)^2}\right|\right] + C_2$$

$$I_2 = 34\,log\left|x - \frac{9}{2} + \sqrt{x^2 - 9x + 20}\right| + C_2$$

Now, putting values of I_1 and I_2 in eq. 1

$$\int \frac{6x+7}{\sqrt{(x-2)(x-4)}} \, . \, dx$$

$$= I_1 + I_2$$

$$= 6\sqrt{x^2 - 9x + 20} + 34\,log\left|x - \frac{9}{2} + \sqrt{x^2 - 9x + 20}\right| + C$$

I S M

Ex 7.4, 23

Integrate $\frac{5x+3}{\sqrt{x^2+4x+10}}$

$$\int \frac{5x+3}{\sqrt{x^2+4x+10}}\, dx$$

Rough

$(x^2+4x+10)' = 2x+4$

$5x + 3 = A(2x+4) + B$

Comparing coefficients of x,

$2A = 5 \Rightarrow A = 5/2$

Comparing constants,

$4A + B = 3 \Rightarrow 4\left(\frac{5}{2}\right) + B = 3 \Rightarrow B = -7$

$$\therefore \int \frac{5x+3}{\sqrt{x^2+4x+10}}\, dx = \frac{5}{2}\underbrace{\int \frac{2x+4}{\sqrt{x^2+4x+10}}\, dx}_{I_1} - 7\underbrace{\int \frac{dx}{\sqrt{x^2+4x+10}}\, dx}_{I_2} \quad ...(1)$$

I S M

Now solving,

$$I_1 = \frac{5}{2}\int \frac{2x+4}{\sqrt{x^2+4x+10}} \, . \, dx$$, we get

$$= \frac{5}{2}\int \frac{1}{\sqrt{t}} \, . \, dt$$

$$I_1 = 5\sqrt{x^2+4x+10} + C_1$$

Solving I_2

$$I_2 = \int \frac{7}{\sqrt{x^2+4x+10}} \, . \, dx$$

$$= 7\int \frac{1}{\sqrt{(x+2)^2+6}} \, . \, dx \; = 7\int \frac{1}{\sqrt{(x+2)^2+(\sqrt{6})^2}} \, . \, dx$$

$$I_2 = 7\,log\left|x+2+\sqrt{x^2+4x+10}\right| + C_2$$

Putting the values of I_1 and I_2 in (1)

$$\int \frac{5x+3}{\sqrt{x^2+4x+10}} \, . \, dx \quad = I_1 - I_2$$

$$= 5\sqrt{x^2+4x+10} + C_1 - 7\,log\left|x+2+\sqrt{x^2+4x+10}\right| + C_2$$

$$= \mathbf{5\sqrt{x^2+4x+10} - 7\,log\left|x+2+\sqrt{x^2+4x+10}\right| + C}$$

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HOME ASSIGNMENT.....

EXERCISE – 7.4

Q. NO : 12, 13, 16, 21, 22.

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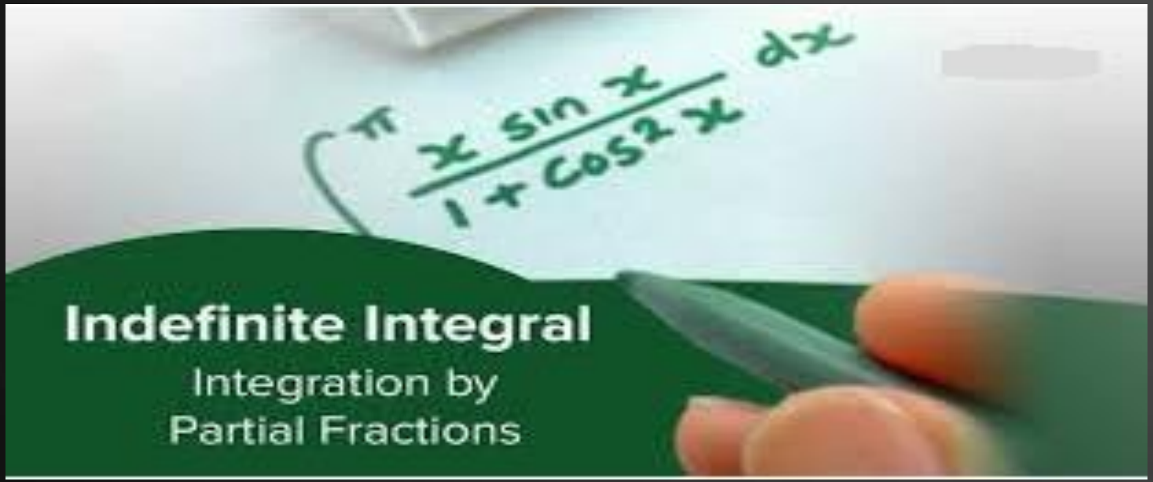
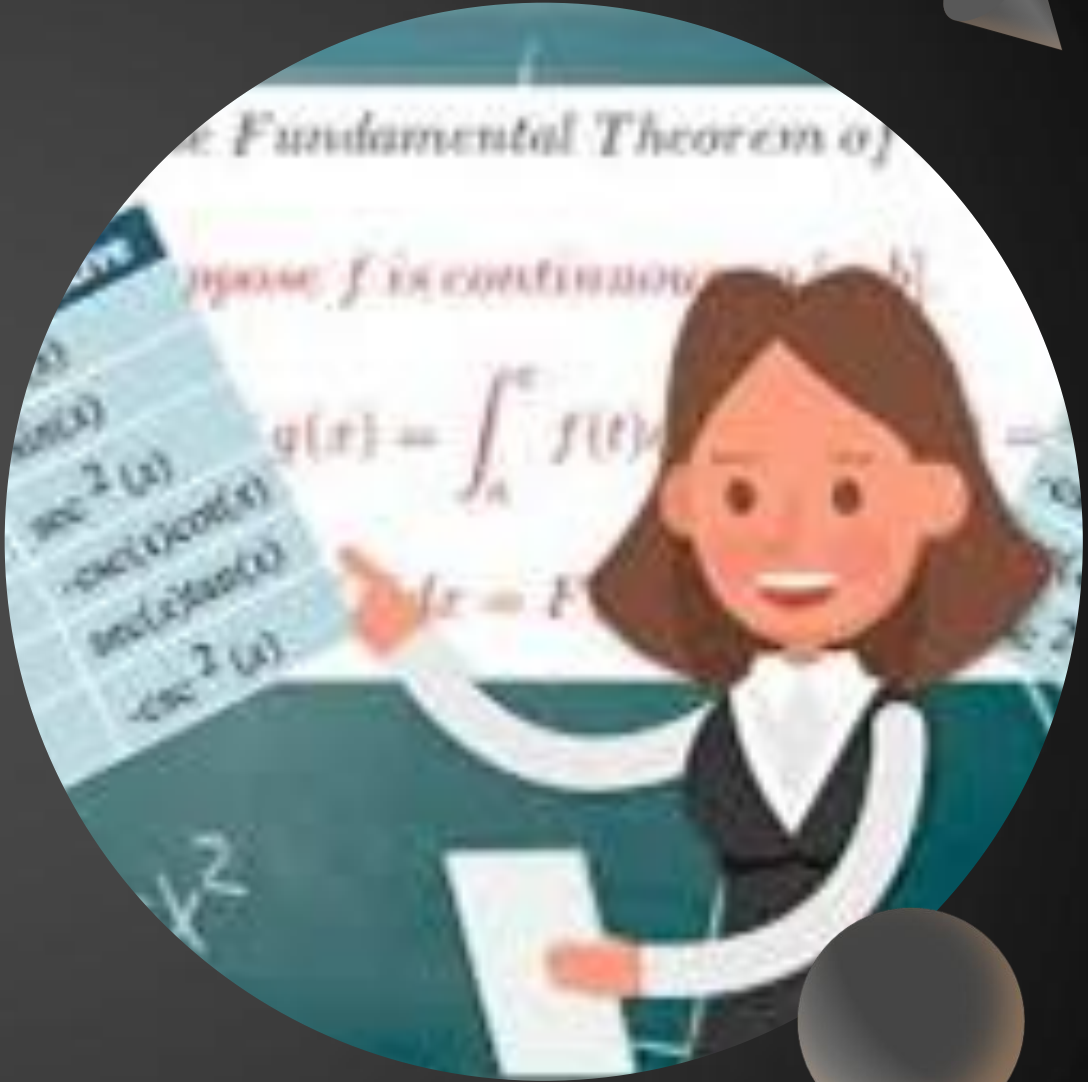
INTEGRALS

MODULE - 6





INTEGRATION BY PARTIAL FRACTIONS

- We know that a rational function is a ratio of two polynomials $\frac{P(x)}{Q(x)}$, where $Q(x) \neq 0$. If the degree of $P(x)$ is less than the degree of $Q(x)$, then it is a **proper** function, otherwise, it is called **improper**.
- Even if a fraction is improper, it can be reduced to a proper fraction by the long division process.
- So, if $\frac{P(x)}{Q(x)}$ is improper, then $\frac{P(x)}{Q(x)} = T(x) + \frac{R(x)}{Q(x)}$, where

  $T(x)$ is a polynomial in x and $\frac{R(x)}{Q(x)}$ is a proper rational function.
- To evaluate $\int \frac{P(x)}{Q(x)}\, dx$, where $\frac{P(x)}{Q(x)}$ is a proper rational function, it is possible to write the integrand as a sum of simpler rational functions by a method called **partial fraction decomposition**.

What are Partial Fractions?

We can do *this* directly:

$$\frac{2}{x-2} + \frac{3}{x+1} \rightarrow \frac{5x-4}{x^2-x-2}$$

... but how do we go in the opposite direction?

$$\frac{2}{x-2} + \frac{3}{x+1} \leftarrow ? \quad \frac{5x-4}{x^2-x-2}$$

Partial Fractions

That is what we are going to discover:

How to find the "parts" that make the single fraction (the "**partial fractions**").

Why Do We Want Them?

Because the partial fractions are each **simpler**.

This can help **integrate** the more complicated fraction.

Partial Fraction Decomposition

When the denominator contains non-repeated linear factors

CASE (i)

Step 1: Factorise the denominator

$$\frac{5x-4}{x^2-x-2} = \frac{5x-4}{(x-2)(x+1)}$$

Step 2: Write one partial fraction for each of those factors

$$\frac{5x-4}{(x-2)(x+1)} = \frac{A}{x-2} + \frac{B}{x+1}$$

Step 3: Multiply through by the denominator so we no longer have fractions

$$5x-4 = A(x+1) + B(x-2)$$

Step 4: Now find the constants A and B

Substituting the roots, or "zeros", of (x−2) and (x+1) can help:

Root for (x+1) is x = -1

$$5(-1) - 4 = A(-1+1) + B(-1-2)$$
$$-9 = 0 + B(-3)$$
$$B = 3$$

Root for (x-2) is x = 2

$$5(2) - 4 = A(2+1) + B(2-2)$$
$$6 = A(3) + 0$$
$$A = 2$$

And we have our answer:

$$\frac{5x-4}{x^2-x-2} = \frac{2}{x-2} + \frac{3}{x+1}$$

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Sometimes you may get a factor with an exponent, like $(x-2)^3$ ...

You need a partial fraction for each exponent from 1 up.

CASE (ii)

When the denominator contains repeated linear factors

Example: $\frac{1}{(x-2)^3}$

Has partial fractions

$$\frac{A}{x-2} + \frac{B}{(x-2)^2} + \frac{C}{(x-2)^3}$$

When you have a quadratic factor you need to include this partial fraction:

$$\frac{Bx + C}{\text{(Your Quadratic)}}$$

When the denominator contains a quadratic factor

Example: $\frac{x^2+15}{(x+3)\ (x^2+3)}$

- Because $(x+3)$ has an exponent of 1, it needs one term A
- And (x^2+3) is a quadratic, so it will need $Bx + C$:

$$\frac{x^2+15}{(x+3)\ (x^2+3)} = \frac{A}{x+3} + \frac{Bx + C}{x^2+3}$$

CASE (iii)

The following table indicates the types of simple partial fractions which can be associated with various rational functions:

S.No	Form of the Rational Function	Form of the Partial Fraction
1.	$\frac{px+q}{(x-a)(x-b)}, a \neq b$	$\frac{A}{x-a} + \frac{B}{x-b}$
2.	$\frac{px+q}{(x-a)^2}$	$\frac{A}{x-a} + \frac{B}{(x-a)^2}$
3.	$\frac{px^2+qx+r}{(x-a)(x-b)(x-c)}$	$\frac{A}{x-a} + \frac{B}{x-b} + \frac{C}{x-c}$
4.	$\frac{px^2+qx+r}{(x-a)^2(x-b)}$	$\frac{A}{x-a} + \frac{B}{(x-a)^2} + \frac{C}{x-b}$
5.	$\frac{px^2+qx+r}{(x-a)(x^2+bx+c)}$	$\frac{A}{x-a} + \frac{Bx+C}{x^2+bx+c}$
	Where $x^2 + bx + c$ cannot be factorised further	

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EX 7.5 Q. 3

Integrate: $\dfrac{3x-1}{(x-1)(x-2)(x-3)}$

We can write the integrand as

$$\frac{3x-1}{(x-1)(x-2)(x-3)} = \frac{A}{(x-1)} + \frac{B}{(x-2)} + \frac{C}{(x-3)}$$

$$= \frac{A(x-2)(x-3) + B(x-1)(x-3) + C(x-1)(x-2)}{(x-1)(x-2)(x-3)}$$

By cancelling denominator

$$3x-1 = A(x-2)(x-3) + B(x-1)(x-3) + C(x-1)(x-2) \quad ...(1)$$

Putting $x = 1$ in (1)

$$2 = A(-1)(-2) + B \times 0 + C \times 0$$

$$2 = 2A \implies A = 1$$

Similarly putting $x = 2$, in (1), we get $B = -5$

Similarly putting x = 3, in (1), we get $C = 4$

Hence we can write it as

$$\int \frac{3x-1}{(x-1)(x-2)(x-3)}\,dx = \int \frac{1}{x-1} + \frac{-5}{x-2} + \frac{4}{x-3}\,dx$$

$$= \int \frac{1}{x-1}\,dx - 5\int \frac{1}{x-2}\,dx + 4\int \frac{1}{x-3}\,dx$$

$$= \log|x-1| - 5\log|x-2| + 4\log|x-3| + C$$

I S M

EX 7.5, Q.12

$\frac{x^3+x+1}{x^2-1}$ (improper fraction)

$$\int \frac{x^3+x+1}{x^2-1} dx = \int \left[x + \frac{2x+1}{x^2-1}\right] dx$$

$$= \int x\, dx + \int \frac{2x}{x^2-1} dx + \int \frac{1}{x^2-1} dx$$

I_1 I_2

$$= \frac{x^2}{2} + \int \frac{dt}{t} + \int \frac{1}{x^2-1} dx$$

$$= \frac{x^2}{2} + \log|t| + \frac{1}{2}\log\left|\frac{x-1}{x+1}\right| + C$$

$$= \frac{x^2}{2} + \log(x^2-1) + \frac{1}{2} \log\left|\frac{x-1}{x+1}\right| + C$$

Rough

x

$x^2 - 1 \overline{)\, x^3 + x + 1}$

$x^3 - x$

(−) (+)

$2x + 1$

To solve I_1

Let $t = x^2 - 1$

$dt = 2x\, dx$

To solve I_2

It is of the form

$$\int \frac{1}{x^2-a^2} dx = \frac{1}{2a} \log\left|\frac{x-a}{x+a}\right|$$

Ex 7.5, 13

$$\frac{2}{(1-x)(1+x^2)}$$

We can write the integrand as $\frac{-2}{(x-1)(1+x^2)}$

Let

$$\frac{-2}{(x-1)(1+x^2)} = \frac{A}{(x-1)} + \frac{Bx+C}{(1+x^2)}$$

$$= \frac{A(1+x^2) + (Bx+C)(x-1)}{(x-1)(1+x^2)}$$

By cancelling denominator

$$-2 = A(1+x^2) + (Bx+C)(x-1) \quad ...(1)$$

Putting x = 1 , in (1)

$$-2 = A(1+1) + (Bx+C)\,0$$

$$\therefore \quad A = -1$$

Putting x = 0 , in (1)

$$-2 = A(1) + C(-1)$$

we get $C = 1$

Equating coefficient of x^2 on both sides of (1)

$0 = A + B \implies B = -A \implies B = 1$

So, we can write

$$\int \frac{-2}{(1-x)(1+x^2)}\,dx = \int \frac{-1}{x-1}\,dx + \int \frac{x+1}{x^2+1}\,dx$$

$$= -\int \frac{1}{x-1}\,dx + \int \frac{x}{x^2+1}\,dx + \int \frac{1}{x^2+1}\,dx$$

↓

I_1

Solving I_1 $= \int \frac{x}{x^2+1}\,dx$

Let $t = x^2 + 1$
$dt = 2x\ dx$

Hence $\int \frac{x}{x^2+1}\,dx = \int \frac{dt}{2(t)} = \frac{1}{2}\log|t| + C_1$

$$= \frac{1}{2}\log|x^2+1| + C_1$$

Therefore

$$\int \frac{2}{(1-x)(1+x^2)}\,dx = \int \frac{-1}{x-1}\,dx + \int \frac{x}{x^2+1}\,dx + \int \frac{1}{x^2+1}\,dx$$

$$= -\log|x-1| + \frac{1}{2}\log|x^2+1| + \tan^{-1}x + C$$

$$= \boldsymbol{-\log|x-1| + \frac{1}{2}\log(x^2+1) + \tan^{-1}x + C}$$

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EX 7.5
Q. 16

Integrate $\dfrac{1}{x(x^n+1)}$

We can write integrand as

$$\frac{1}{x(x^n+1)} = \frac{1}{x(x^n+1)} \times \frac{x^{n-1}}{x^{n-1}} \quad \text{(multiply numerator and denominator by } x^{n-1}$$

$$= \frac{x^{n-1}}{x^n(x^n+1)}$$

Let t = x^n

$dt = n\,x^{n-1}\,dx$

Therefore $\int \frac{x^{n-1}}{x^n(x^n+1)}\,dx = \frac{1}{n}\int \frac{dt}{t(t+1)}$

We can write the integrand as

$$\frac{1}{t(t+1)} = \frac{A}{t} + \frac{B}{t+1}$$

By cancelling denominator

$$1 = A(t+1) + Bt \qquad ...(1)$$

Putting t = 0 in (1) $A = 1$

Similarly putting t = −1 in (1) $B = -1$

Thus, $\int \frac{dt}{t(t+1)} = \int \frac{1\,dt}{t} - \int \frac{-1}{t+1}\,dt = \log|t| - \log|t+1| + C$

$$= \log\left|\frac{t}{t+1}\right| + C$$

$$\int \frac{1}{x(x^n+1)} = \frac{1}{n}\log\left|\frac{x^n}{x^n+1}\right| + C$$

I S M

Ex 7.5, 18

Integrate the function $\frac{(x^2+1)(x^2+2)}{(x^2+3)(x^2+4)}$

(Improper fraction)

Let t = x^2

$$\frac{(x^2+1)(x^2+2)}{(x^2+3)(x^2+4)} = \frac{(t+1)(t+2)}{(t+3)(t+4)}$$

$$= \frac{t^2+3t+2}{t^2+7t+12}$$

$$= 1 + \frac{-4t-10}{t^2+7t+12}$$

$$= 1 - \frac{(4t+10)}{(t+3)(t+4)}$$

Rough

$$\begin{array}{r|l} & \quad 1 \\ t^2+7t+12 & t^2+3t+2 \\ & t^2+7t+12 \\ & (-)\ (-)\ (-) \\ \hline & -4t-10 \end{array}$$

We can write $\frac{4t+10}{(t+3)\,(t+4)} = \frac{A}{(t+3)} + \frac{B}{(t+4)}$

$$4t + 10 = A(t+4) + B(t+3) \quad ...(1)$$

Putting t = − 4 in (1), we get $B = 6$

Putting t = −3 in (1), we get $A = -2$

Hence we can write $\dfrac{4t+10}{(t+3)\,(t+4)} = \dfrac{-2}{(t+3)} + \dfrac{6}{(t+4)}$

Therefore

$$\int \frac{(x^2+1)(x^2+2)}{(x^2+3)(x^2+4)}\,dx = \int 1\,dx - \left[\frac{-2}{(x^2+3)} + \frac{6}{(x^2+4)}\right]dx$$

$$= \int 1.dx + \int \frac{2}{(x^2+3)}\,dx - \int \frac{6}{(x^2+4)}\,dx$$

$$= \int 1.dx + 2\int \frac{1}{x^2+(\sqrt{3})^2}\,dx - 6\int \frac{1}{(x^2+2^2)}\,dx$$

$$= x + 2 \times \frac{1}{\sqrt{3}}\tan^{-1}\frac{x}{\sqrt{3}} - 6 \times \frac{1}{2}\tan^{-1}\frac{x}{2} + C$$

$$= \boldsymbol{x + \frac{2}{\sqrt{3}}\,tan^{-1}\left(\frac{x}{\sqrt{3}}\right) - 3\,tan^{-1}\left(\frac{x}{2}\right) + C}$$

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EX 7.5, Q.19

$$\frac{2x}{(x^2+1)(x^2+3)}$$

Let $x^2 = t$

$2x\,dx = dt$

Can you suggest any other alternate method to solve this sum…?

$$\int \frac{2x}{(x^2+1)(x^2+3)}\,dx = \int \frac{dt}{(t+1)(t+3)}$$

$$= \frac{1}{2}\int \frac{(t+3)-(t+1)}{(t+1)(t+3)}dt$$ [multiplying and dividing by 2]

$$= \frac{1}{2}\int \left(\frac{1}{t+1} - \frac{1}{t+3}\right) dt$$

$$= \frac{1}{2}\left[\int \frac{1}{t+1}\,dt - \int \frac{1}{t+3}\,dt\right]$$

$$= \frac{1}{2}[log|t+1| - log|t+3|] + C$$

$$= \frac{1}{2}\left[\log\left|\frac{t+1}{t+3}\right|\right] + C$$

$$= \frac{1}{2}\, log\left[\frac{x^2+1}{x^2+3}\right] + C$$

EX 7.5, Q.21

$$\frac{1}{e^x-1}$$

Let $e^x = t \Rightarrow e^x - 1 = t - 1$

$e^x\, dx = dt \Rightarrow dx = \frac{dt}{t}$

$$\int \frac{1}{e^x-1}\, dx = \int \frac{1}{t} \times \frac{dt}{t-1} = \int \frac{dt}{t(t-1)}$$

$$= \int \frac{t-(t-1)}{t(t-1)} dt$$

$$= \int \left(\frac{1}{t-1} - \frac{1}{t}\right) dt$$

$$= \left[\int \frac{1}{t-1}\, dt - \int \frac{1}{t}\, dt\right]$$

$$= [log|t-1| - log|t|] + C$$

$$= \left[\log\left|\frac{t-1}{t}\right|\right] + C$$

$$= \boldsymbol{log\left[\frac{e^x-1}{e^x}\right] + C}$$

Can partial fractions be used to solve this….?

HOME ASSIGNMENT

EXERCISE 7.5 – Q. NO: 4, 5, 6, 8, 10, 11, 15, 17

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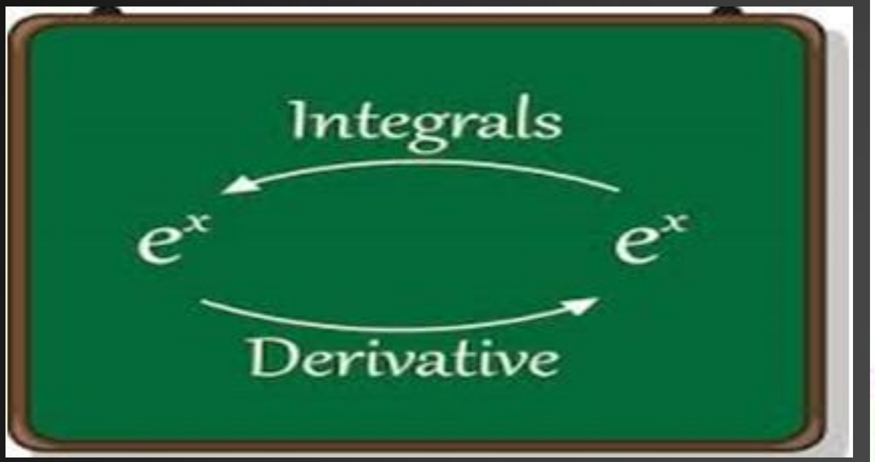
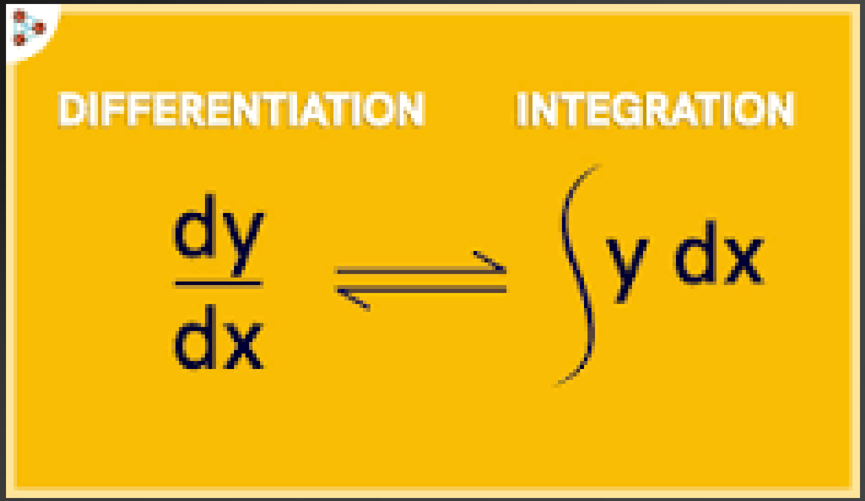
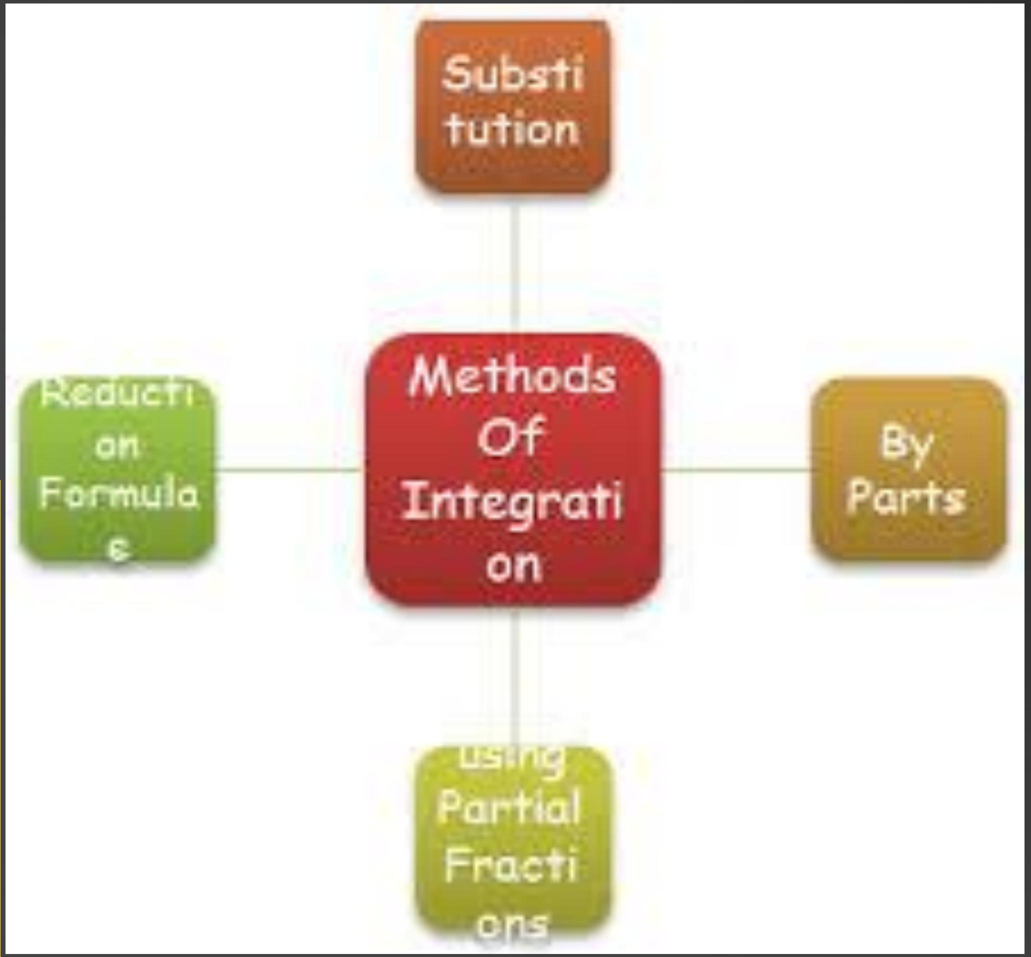
INTEGRALS

MODULE - 7







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Integration by Parts

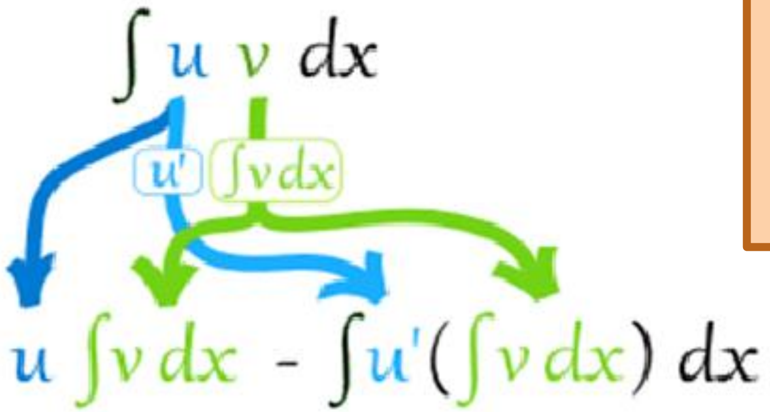
Integration by Parts is a special method of integration that is often useful when two functions are multiplied together .

let us see the rule:

$$\int u\,v\,dx = u\int v\,dx - \int u'\left(\int v\,dx\right)dx$$

- **u** is the first function u(x)
- **v** is the second function v(x)
- **u'** is the derivative of the function u(x)

As a diagram:



The formula can be stated as:

"The **integral** of the **product of two functions** = (**first function**) x (**integral** of the **second function**) - [**Integral** of (**derivative** of **first function**) x (**integral** of the **second function**)]"

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Where Did "Integration by Parts" Come From?

It is based on the Product Rule for Derivatives:

$$(f(x).g(x))' = f(x).g'(x) + g(x).f'(x)$$

Integrate both sides and rearrange:

$$\int (f(x).g(x))'\,dx = \int f(x).g'(x)\,dx + \int g(x).f'(x)\,dx$$

$$f(x).g(x) = \int f(x).g'(x)\,dx + \int g(x).f'(x)\,dx$$

$$\int f(x).g'(x)\,dx = f(x).g(x) - \int g(x).f'(x)\,dx$$

Let $f(x) = u$ and $g'(x) = v$. Then, $f'(x) = u'$ and $g(x) = \int v\,dx$

$$\int uv\,dx = u\int v\,dx - \int u'\left(\int v\,dx\right)dx$$

I S M

Let's look at an example:

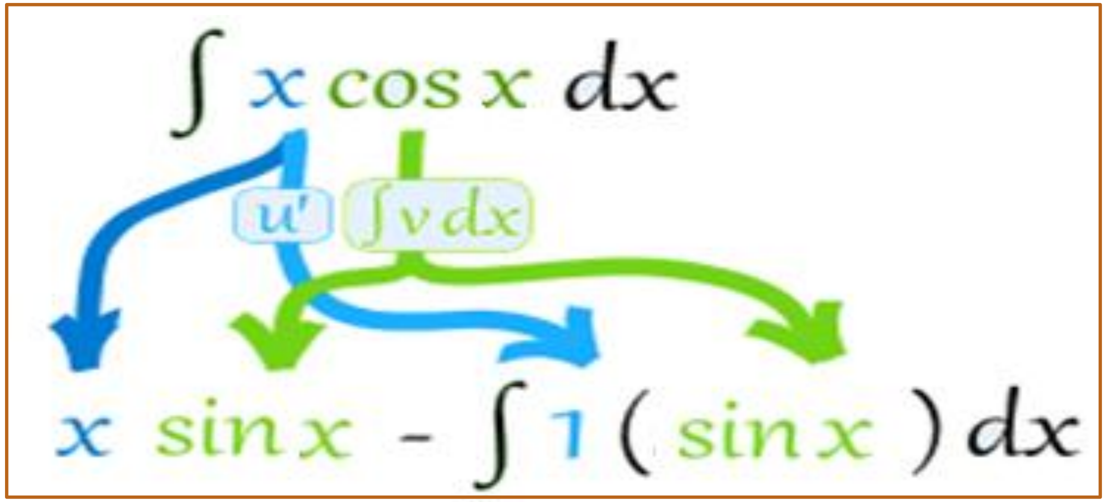
What is $\int x\, cosx\, dx$?

First choose which functions for u and v $\longrightarrow$ $u = x$ and $v = \cos x$.

Now it is in the format $\int u\, v\, dx$, so we can proceed :

Differentiate u : $u' = \frac{d}{dx}(x) = 1$

Integrate v : $\int v\, dx = \int \cos x\, dx = \sin x$



Simplify and solve :

$x \sin x - \int sinx\, dx$

$x \sin x + \cos x + C$

DOES INTERCHANGING OF U AND V MAKES A DIFFERENCE :

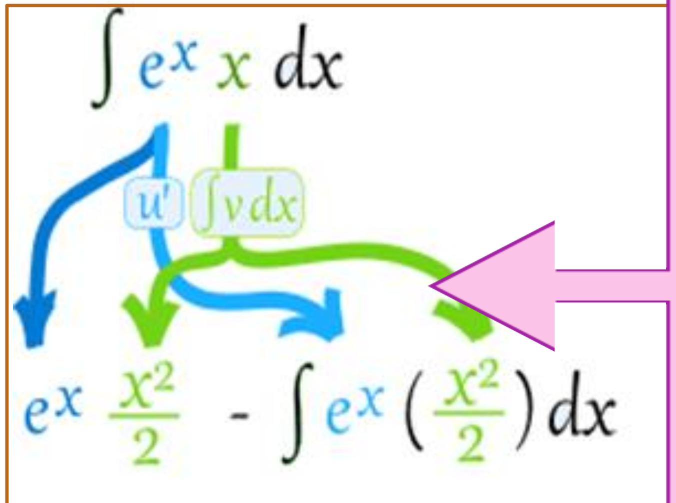
- What is $\int e^x \, x \, dx$?

Choose u and $v \longrightarrow$ $u = e^x$

$v = x$

Differentiate u : $u' = \frac{d}{dx}(e^x) = e^x$

Integrate v : $\int v \, dx = \int x \, dx = \frac{x^2}{2}$



Next step will yield $\int e^x \frac{x^3}{3} dx$ and every subsequent step will keep on getting higher powers of x and hence integration will never terminate.

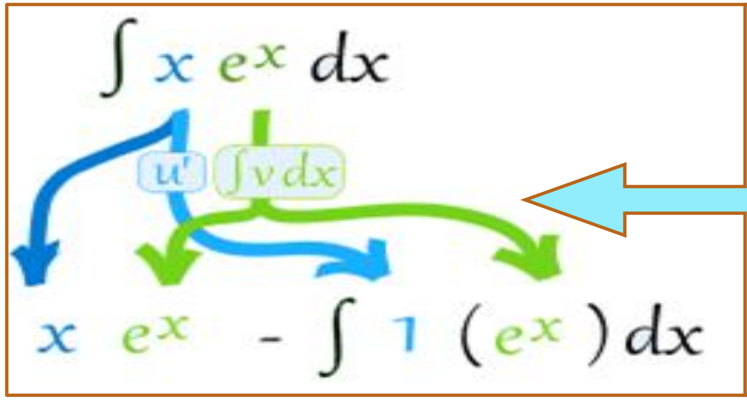
When will it end

- **Maybe we could choose u and v differently .....**

Choose $u = x$ and $v = e^x$

Differentiate u : $u' = \frac{d}{dx}(x) = 1$

Integrate v : $\int v \, dx = \int e^x \, dx = e^x$

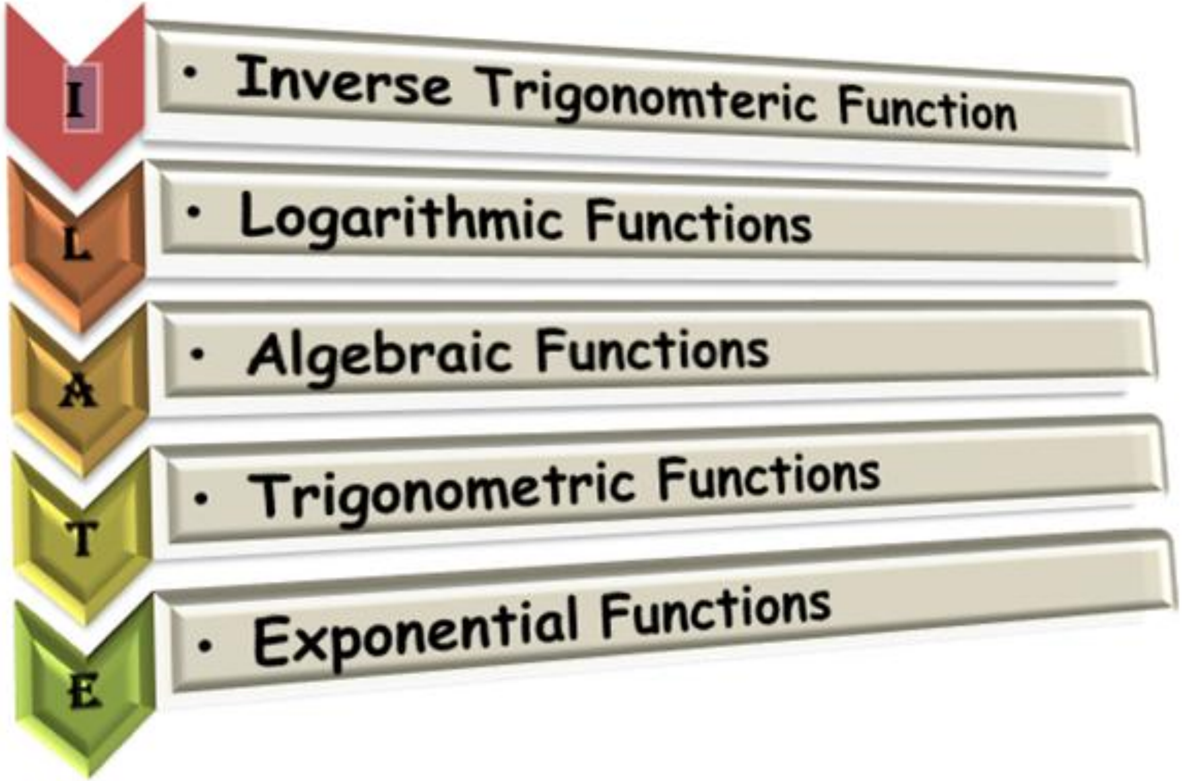


By interchanging the choice of u and v the last integration becomes much simpler

Moral $\longrightarrow$ Choose u and v carefully !!!

Choose u that gets simpler when you differentiate it and v that doesn't get complicated when you integrate it.

The chart given below illustrates the preference order generally adopted for the selection of the first function:



- **I**: Inverse trigonometric functions such as $\sin^{-1}(x)$, $\cos^{-1}(x)$, $\tan^{-1}(x)$
- **L**: Logarithmic functions such as $\log(x)$
- **A**: Algebraic functions such as x^2, x^3
- **T**: Trigonometric functions such as $\sin(x)$, $\cos(x)$, $\tan(x)$
- **E**: Exponential functions such as e^x, 3^x

ISM

Ex 7.6, 8

$x\tan^{-1}x$

$\int x\tan^{-1}x\ dx$

Algebraic ↙ ↘ Inverse

We know that

$\int u\,v\ dx = u\int v\ dx - \int\left(u'\int v\ dx\right)dx$

Putting $u = \tan^{-1}x$ and $v = x$

$\int x\tan^{-1}x\ dx = \int(\tan^{-1}x)\,x\,dx$

$= \tan^{-1}x\int x\,dx - \int\left(\frac{d(\tan^{-1}x)}{dx}\int x\,.dx\right)dx$

$= \tan^{-1}x\,.\,\frac{x^2}{2} - \int\frac{1}{1+x^2}\,.\,\frac{x^2}{2}\,.\,dx$

$= \frac{x^2}{2}\tan^{-1}x - \frac{1}{2}\int\frac{x^2}{x^2+1}\ dx$

$= \frac{x^2}{2}\tan^{-1}x - \frac{1}{2}\int\frac{x^2+1-1}{x^2+1}\ dx$

$= \frac{x^2}{2}\tan^{-1}x - \frac{1}{2}\left[\int\frac{x^2+1}{x^2+1}\ dx - \int\frac{dx}{x^2+1}\right]$

$= \frac{x^2}{2}\tan^{-1}x - \frac{1}{2}\left[\int 1\ dx - \int\frac{dx}{x^2+1}\right]$

Using

$\int\frac{dx}{a^2+x^2} = \frac{1}{a}\tan^{-1}\frac{x}{a} + C$

$= \frac{x^2}{2}\tan^{-1}x - \frac{1}{2}x + \frac{1}{2}\times\frac{1}{1}\tan^{-1}\frac{x}{1} + C$

Example 18

Find $\int \log x \, d\,x$

Here, we are unable to guess a function whose derivative is *log x*.
Hence, we take log x as the first function and constant function 1 as the second function.

$$\int \log x \;\; dx$$

$$= \int (\log x) \,.\, 1 \; dx$$

Using by parts

I^{st} function $u = \log x$ & II^{nd} function $v = 1$

We know that

$$\int u\, v \;\; dx = u \int v \;\; dx - \int (u' \int v \;\; dx) \, dx$$

$$\int (\log x) \,.\, 1 \; dx = \log x \int 1 \,.\, dx - \int \left(\frac{d(\log x)}{dx} \int 1 \,.\, dx \right) dx$$

$$= (\log x) x - \int \frac{1}{x} \,.\; x \;.\, dx$$

$$= x \log x - \int 1 \,.\, dx$$

$$= \boldsymbol{x \log x - x + C}$$

I S M

Example 20 : Find $\int \frac{x \sin^{-1} x}{\sqrt{1-x^2}}\,dx$

Method -1 (Directly use product rule)

$$\int \frac{x \sin^{-1} x}{\sqrt{1-x^2}}\,dx = \sin^{-1} x \quad \frac{x}{\sqrt{1-x^2}}$$

Inverse Algebraic

To find $\int \frac{x\,dx}{\sqrt{1-x^2}}$

Let $1-x^2 = t$

Then, $-2x\,dx = dt$

$$\int \frac{x\,dx}{\sqrt{1-x^2}} = \frac{-1}{2}\int \frac{dt}{\sqrt{t}} = -\sqrt{t} = -\sqrt{1-x^2}$$

Hence,

$$\int \frac{x \sin^{-1} x}{\sqrt{1-x^2}}\,dx$$

$$= \sin^{-1} x\,(-\sqrt{1-x^2}) - \int \frac{1}{\sqrt{1-x^2}}(-\sqrt{1-x^2})\,dx$$

$$= -\sqrt{1-x^2}\,\sin^{-1} x + x + C$$

$$= x - \sqrt{1-x^2}\,\sin^{-1} x + C$$

Method -2 (Substitution and then by parts)

If $\sin^{-1} x = t \implies x = \sin t$

then $\frac{dx}{\sqrt{1-x^2}} = dt$

$$\int \frac{x \sin^{-1} x}{\sqrt{1-x^2}}\,dx = \int (x)(\sin^{-1} x)\frac{dx}{\sqrt{1-x^2}}$$

$$= \int (\sin t)\,t\,dt$$

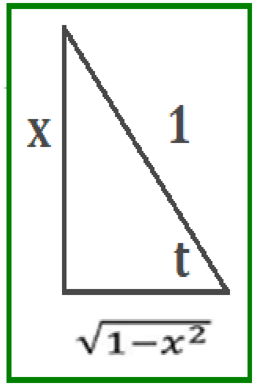
$$= t.\int(\sin t)\,dt - \int\left[\left(\frac{d}{dt}t\right).\int(\sin t)\,dt\right]dt$$

$$= t.(-\cos t) - \int 1.(-\cos t)\,dt$$

$$= t.(-\cos t) + \sin t + c$$

$$= \sin^{-1} x\,(-\sqrt{1-x^2}) + x + c$$

$$= x - \sqrt{1-x^2}\,\sin^{-1} x + c$$



Ex 7.6, 14

$x(\log x)^2$

$\int x \quad (\log x)^2 . dx$

Algebraic ↙ ↘ Logarithmic

$\therefore \int x(\log x)^2 . dx = \int (\log x)^2 \, x \, . dx$

$= (\log x)^2 \int x . dx - \int \left(\frac{d(\log x)^2}{dx} \int x . dx \right) dx$

$= (\log x)^2 . \frac{x^2}{2} - \int \left(2(\log x) \frac{1}{x} \int x . dx \right) dx$

$= \frac{x^2}{2} (\log x)^2 - 2 \int \frac{\log x}{x} . \frac{x^2}{2} \, dx$

$= \frac{x^2}{2} (\log x)^2 - \underbrace{\int x \log x \; dx}_{I_1}$ ...(1)

I S M

Solving I_1 $= \int x \ \log x \ dx$

Algebraic (x) **Logarithmic** ($\log x$)

$$\int x \log x \ dx = \int (\log x) x \, dx$$

$$= \log x \int x \, dx - \int \left(\frac{d(\log x)}{dx} \int x . dx \right) dx$$

$$= \log x \left(\frac{x^2}{2} \right) - \int \frac{1}{x} . \frac{x^2}{2} . dx$$

$$= \frac{x^2}{2} \log x - \frac{1}{2} \int x . dx$$

$$= \frac{x^2}{2} \log x - \frac{1}{2} . \frac{x^2}{2} + C$$

$$= \frac{x^2}{2} \ log \ x - \frac{x^2}{4} + C$$

Putting value of I_1 in (1),

$$\frac{x^2}{2} (\log x)^2 - \int \boldsymbol{x . log \ x \ dx} = \frac{x^2}{2} (\log x)^2 - \left(\frac{x^2 (\log x)}{2} - \frac{x^2}{4} + C \right)$$

$$= \frac{x^2}{2} (\log x)^2 - \frac{x^2 (\log x)}{2} + \frac{x^2}{4} - C$$

$$= \boldsymbol{\frac{x^2}{2} (log \ x)^2 - \frac{x^2 (log \ x)}{2} + \frac{x^2}{4} + C}$$

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Integral of the type : $\int e^x\,[f(x) + f'(x)]\,dx$

Solving $I_1 = \int e^x \; f(x)\,dx$

Exponential (for e^x) — Algebraic (for $f(x)$)

Let

$$I = \int e^x\,[f(x) + f'(x)]\,dx = \int e^x f(x)\,dx + \int e^x f'(x)\,dx$$

$$= \quad I_1 \quad + \int e^x f'(x)\,dx$$

$$= \left[f(x)\,e^x - \int f'(x)\,e^x\,dx\right] + \int e^x f'(x)\,dx$$

$$\therefore \int e^x\,[f(x) + f'(x)]\,dx = \int e^x\,f(x)\,dx + C$$

Ex 7.6, 18

Integrate the function : $e^x\left(\frac{1+\sin x}{1+\cos x}\right)$

Simplifying function $e^x\left(\frac{1+\sin x}{1+\cos x}\right) = e^x\left(\left(\frac{1}{1+\cos x}\right)+\left(\frac{\sin x}{1+\cos x}\right)\right)$

$$= e^x\left(\left(\frac{1}{2cos^2\frac{x}{2}}\right)+\left(\frac{2\sin\left(\frac{x}{2}\right)\cos\left(\frac{x}{2}\right)}{2\,cos^2\left(\frac{x}{2}\right)}\right)\right)$$

$$= e^x\left(\frac{1}{2}\,.\,\sec^2\frac{x}{2} + \tan\left(\frac{x}{2}\right)\right)$$

$$= e^x\left(\tan\left(\frac{x}{2}\right)+\frac{1}{2}\sec^2\left(\frac{x}{2}\right)\right)$$

It is of the form

$$\int e^x[f(x)+f'(x)]\,dx = e^x f(x) + C$$

Thus,

Our Integration becomes $\int e^x\left(\frac{1+\sin x}{1+\cos x}\right)dx$ $= \boldsymbol{e^x \tan\frac{x}{2} + C}$

I S M

Example 22

Find $\int \frac{(x^2+1)e^x}{(x+1)^2} dx$

$$\int \frac{x^2+1}{(x+1)^2} . e^x \, dx = \int \frac{x^2+1+1-1}{(x+1)^2} . e^x . dx \quad \left[\textit{Adding and subtracting 1 in numerator} \right]$$

$$= \int \left[\frac{x^2-1}{(x+1)^2} + \frac{2}{(x+1)^2} \right] e^x dx$$

$$= \int e^x \left[\frac{x-1}{x+1} + \frac{2}{(x+1)^2} \right] dx$$

It is of form

$$\int e^x [f(x) + f'(x)] \, dx = e^x f(x) + C$$

Where $f(x) = \frac{x-1}{x+1}$

$$f'(x) = \frac{d}{dx} \left[\frac{x-1}{x+1} \right] = \frac{2}{(x+1)^2}$$

Thus,

$$\int \frac{x^2+1}{(x+1)^2} . e^x = \int e^x \left[\frac{x-1}{x+1} + \frac{2}{(x+1)^2} \right] dx$$

$$= e^x \left[\frac{x-1}{x+1} \right] + C$$

Some more special types of standard Integrals…..

(i) $\int \sqrt{x^2 - a^2}\, dx = \frac{x}{2}\sqrt{x^2 - a^2} - \frac{a^2}{2} \log\left|x + \sqrt{x^2 - a^2}\right| + C$

(ii) $\int \sqrt{x^2 + a^2}\, dx = \frac{x}{2}\sqrt{x^2 + a^2} + \frac{a^2}{2} \log\left|x + \sqrt{x^2 + a^2}\right| + C$

(iii) $\int \sqrt{a^2 - x^2}\, dx = \frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2} \sin^{-1}\frac{x}{a} + C$

The above integrals can be proved by taking the constant function 1 as the second function and integrating by parts.

Example 23

Find $\int \sqrt{x^2+2x+5}\ dx$

$$\int \sqrt{x^2+2x+5}\ dx = \int \sqrt{x^2+2x+1+4}\ dx$$

$$= \int \sqrt{(x+1)^2+4}\ dx$$

It is of the form

$$\int \sqrt{x^2+a^2}\,dx = \frac{x}{2}\sqrt{x^2+a^2} + \frac{a^2}{2} log\left|x+\sqrt{x^2+a^2}\right| + C$$

$$= \frac{x+1}{2}\sqrt{(x+1)^2+4} + 2\,log\left|x+1+\sqrt{(x+1)^2+4}\right| + C$$

$$= \frac{1}{2}(x+1)\sqrt{x^2+2x+5} + 2\log\left|x+1+\sqrt{x^2+2x+5}\right| + C$$

Example 24

Find $\int \sqrt{3-2x-x^2}\ dx$

$$\int \sqrt{3-2x-x^2}\ dx = \int \sqrt{3-(2x+x^2)}\,dx$$

$$= \int \sqrt{4-(x^2+2x+1^2)}\ dx$$ (Adding and Subtracting 1)

$$= \int \sqrt{2^2-(x+1)^2}\ dx$$

It is of the form

$$\int \sqrt{a^2-x^2}\ dx = \frac{1}{2}x\sqrt{a^2-x^2} + \frac{a^2}{2} sin^{-1}\frac{x}{a} + C$$

$$= \frac{1}{2}(x+1)\sqrt{2^2-(x+1)^2} + \frac{2^2}{2} sin^{-1}\frac{(x+1)}{2} + C$$

$$= \frac{1}{2}(x+1)\sqrt{3-2x-x^2} + 2\,sin^{-1}\frac{(x+1)}{2} + C$$

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HOME ASSIGNMENT……

- **COMPLETE EX – 7.6 AND EX – 7.7**

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